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# Electro–magneto-encephalography for a three-shell model: distributed current in arbitrary, spherical and ellipsoidal geometries

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The problem of determining a continuously distributed neuronal current inside the brain under the assumption of a three-shell model is analysed. It is shown that for an arbitrary geometry, electroencephalography (EEG) provides information about one of the three functions specifying the three components of the current, whereas magnetoencephalography (MEG) provides information about a combination of this function and of one of the remaining two functions. Hence, the simultaneous use of EEG and MEG yields information about two of the three functions needed for the reconstruction of the current. In particular, for spherical and ellipsoidal geometries, it is possible to determine the angular parts of these two functions as well as to obtain an explicit constraint satisfied by their radial parts. The complete determination of the radial parts, as well as the determination of the third function, requires some additional *a priori* assumption about the current. One such assumption involving harmonicity is briefly discussed.

**Keywords:** electroencephalography; magnetoencephalography; brain imaging;  
inverse problems

## 1. INTRODUCTION

The medical significance of electroencephalography (EEG) and magnetoencephalography (MEG) is well established (Ribary *et al.* 1991; Hauk *et al.* 2001; Papanicolaou 2006; Langheim *et al.* 2006). However, the lack of uniqueness of the solution of the associated inverse problems remains a challenging problem. In particular, the non-uniqueness of the inverse problem is considered as the Achilles' heel of MEG. In this context, a complete answer to the non-uniqueness question for a homogeneous *spherical* model was presented in Fokas *et al.* (1996, 2004) where it was shown that: (i) the only part of a continuously distributed current that can be reconstructed via MEG consists of certain moments of one of the two functions specifying the tangential component of the current (the other function specifying the tangential component, as well as the radial component of the current, is ‘invisible’ in the spherical model of MEG) and (ii) it is possible to reconstruct uniquely the current that minimizes the  $L_2$ -norm. Some of these results were extended, from a spherical to a *star-shaped* geometry in Dassios *et al.* (2005). Although analogous results for EEG have not been obtained so far, the mathematical notion of complementarity of MEG and EEG for a spherical geometry was introduced in Dassios *et al.* (2007a) where, by expanding the neuronal current in terms of *vector* spherical harmonics, the

following results were obtained: (i) the component of a continuously distributed neuronal current that generates the electric potential (and hence measured by EEG) lives in the orthogonal complement of the component of the current that generates the magnetic potential (which is measured by MEG) and (ii) EEG and MEG measurements can be used to specify the angular dependence of these components as well as certain constraints about the associated radial dependence.

In this paper, a straightforward approach for the solution of the inverse problem for both EEG and MEG is introduced. This approach, which is much simpler than the one used in Fokas *et al.* (1996, 2004), yields a complete answer to the non-uniqueness question even in the case of an arbitrary geometry. Furthermore, in the particular cases of spherical and ellipsoidal geometries, it yields effective formulae for the ‘visible’ component of the current.

The analysis presented here is concerned with a *continuously* distributed current; the opposite case where the current is *localized in a finite number of points*, i.e. the case of a collection of dipoles, is analysed in Dassios & Fokas (preprint *a,b*) for spherical and ellipsoidal geometries, respectively. For other related important works, see El Badia & Ha-Duong (2000), Jerbi *et al.* (2002), Nara & Ando (2003), Nolte & Dassios (2005), Albanese & Monk (2006), Peng *et al.* (2006), Nara *et al.* (2007) and Leblond *et al.* (preprint).

This paper is organized as follows: the equations needed for EEG and MEG in a three-shell model are

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derived in §2; this is done for the sake of completeness so that this paper is self-contained. The inverse problems for EEG and MEG for an arbitrary geometry are analysed in §3. The particular cases of spherical and ellipsoidal geometries are considered in §§4 and 5, respectively. In §6, these results are discussed further and a possible constraint that can lead to a unique current is mentioned.

### 1.1. Notations

- $\mathbf{J}^p(\mathbf{r})$ ,  $\mathbf{B}(\mathbf{r})$ ,  $\mathbf{E}(\mathbf{r})$ ,  $U(\mathbf{r})$  and  $u(\mathbf{r})$  will denote the neuronal current (primary current), the magnetic field, the electric field, the magnetic potential and the electric potential, respectively, at the point  $\mathbf{r} \in \mathbb{R}^3$ .
- $\sigma$  and  $\mu$  denote conductivity and permeability.
- $\mathcal{Q}$  will denote the three-dimensional space occupied by the conducting medium and  $\partial\mathcal{Q}$  its boundary. The subscripts  $c$ ,  $f$ ,  $b$  and  $s$  will denote brain (cerebrum), fluid, bone and scalp, respectively.  $\mathcal{Q}_e$  will denote the space outside the head (exterior space).
- $\tau$  and  $\mathbf{Q}(\tau)$  will denote the position and the moment of a single dipole,  $\tau \in \mathcal{Q}_c$ .
- The ‘hat’ symbol on top of a vector will denote that this vector has unit length. In particular,  $\hat{\mathbf{n}}$  denotes the unit outward normal to the surface  $\partial\mathcal{Q}$ . The derivative  $\partial/\partial n$  will denote differentiation along the direction of  $\hat{\mathbf{n}}$ .
- $dV(\mathbf{r}')$  and  $dS(\mathbf{r}')$  will denote the volume and the surface differentials associated with  $\mathcal{Q}$  and  $\partial\mathcal{Q}$ .
- The spherical coordinates of the point  $\tau$  will be denoted by  $(\tau, \theta, \varphi)$ , where

$$0 \leq \tau < \alpha, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi < 2\pi$$

and the spherical coordinates of the point  $\mathbf{r}$  will be denoted by  $(r, \Theta, \Phi)$ . The ellipsoidal coordinates of the point  $\tau$  will be denoted by  $(\rho, \mu, \nu)$ , where

$$h_2 \leq \rho < c_1, \quad h_2 \leq \mu \leq h_3, \quad -h_3 \leq \nu \leq h_3$$

and the ellipsoidal coordinates of the point  $\mathbf{r}$  will be denoted by  $(R, M, N)$ .

## 2. THE BASIC EQUATIONS

Electromagnetic activity as measured by EEG and MEG is governed by the quasi-static reduction of Maxwell’s equations (Plonsey & Heppner 1967) formulated in a conducting space  $\mathcal{Q}$

$$\nabla \times \mathbf{E} = \mathbf{0}, \quad \nabla \times \mathbf{B} = \mu(\mathbf{J}^p + \sigma \mathbf{E}), \quad \nabla \cdot \mathbf{B} = 0. \quad (2.1)$$

The first of these equations implies the existence of a function  $u$  (the electric potential) such that

$$\mathbf{E} = -\nabla u, \quad (2.2)$$

whereas the second and the third of equations (2.1) imply the compatibility condition

$$\nabla \cdot \mathbf{J}^p + \sigma \nabla \cdot \mathbf{E} = 0. \quad (2.3)$$

A well-known model for the electromagnetic activity of the head consists of a space  $\mathcal{Q}_c$  modelling the space occupied by the cerebrum, surrounded by three shells  $\mathcal{Q}_f$ ,  $\mathcal{Q}_b$ ,  $\mathcal{Q}_s$ , modelling the spaces occupied by the cerebrospinal fluid, the skull and the skin. These

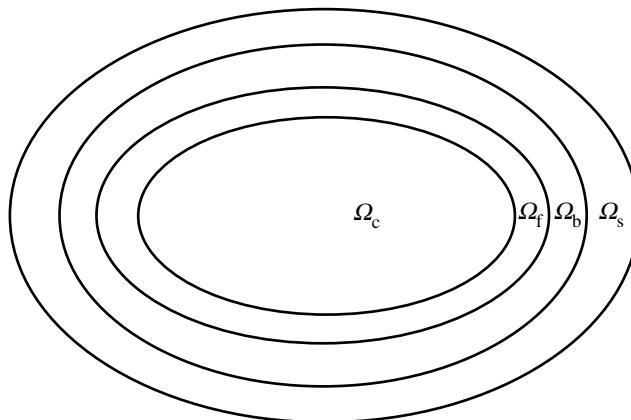


Figure 1. The space  $\mathcal{Q}_c$  (cerebrum), bounded by  $\partial\mathcal{Q}_c$ , the space  $\mathcal{Q}_f$  (fluid) by  $(\partial\mathcal{Q}_c, \partial\mathcal{Q}_f)$ , the space  $\mathcal{Q}_b$  (bone) by  $(\partial\mathcal{Q}_f, \partial\mathcal{Q}_b)$  and the space  $\mathcal{Q}_s$  (skin) by  $(\partial\mathcal{Q}_b, \partial\mathcal{Q}_s)$ .

compartments are distinguished by their different values of electric conductivity, which will be denoted respectively by  $\sigma_c$ ,  $\sigma_f$ ,  $\sigma_b$  and  $\sigma_s$ . The spaces  $\mathcal{Q}_c$ ,  $\mathcal{Q}_f$ ,  $\mathcal{Q}_b$  and  $\mathcal{Q}_s$  are bounded by the surfaces  $\partial\mathcal{Q}_c$ ,  $(\partial\mathcal{Q}_c, \partial\mathcal{Q}_f)$ ,  $(\partial\mathcal{Q}_f, \partial\mathcal{Q}_b)$  and  $(\partial\mathcal{Q}_b, \partial\mathcal{Q}_s)$ , respectively (figure 1).

The primary current  $\mathbf{J}^p$  has support only in  $\mathcal{Q}_c$ . Hence replacing in equation (2.3)  $\mathbf{E}$  by  $-\nabla u_c$  and using the continuity of the normal component of the electric field across  $\partial\mathcal{Q}_c$ , it follows that  $u_c$  satisfies

$$\left. \begin{aligned} \sigma_c \Delta u_c(\mathbf{r}) &= \nabla \cdot \mathbf{J}^p(\mathbf{r}), & \mathbf{r} \in \mathcal{Q}_c \\ \sigma_c \frac{\partial u_c(\mathbf{r})}{\partial n} &= \sigma_f \frac{\partial u_f(\mathbf{r})}{\partial n}, & \mathbf{r} \in \partial\mathcal{Q}_c. \end{aligned} \right\} \quad (2.4)$$

Similarly, employing equations (2.2) and (2.3) and using the continuity of the electric field across  $\partial\mathcal{Q}_f$  and  $\partial\mathcal{Q}_b$ , it follows that  $u_f$ ,  $u_b$  and  $u_s$  satisfy

$$\left. \begin{aligned} \Delta u_f(\mathbf{r}) &= 0, & \mathbf{r} \in \mathcal{Q}_f; \quad u_f(\mathbf{r}) = u_c(\mathbf{r}), & \mathbf{r} \in \partial\mathcal{Q}_c \\ \sigma_f \frac{\partial u_f(\mathbf{r})}{\partial n} &= \sigma_b \frac{\partial u_b(\mathbf{r})}{\partial n}, & \mathbf{r} \in \partial\mathcal{Q}_f, \end{aligned} \right\} \quad (2.5)$$

$$\left. \begin{aligned} \Delta u_b(\mathbf{r}) &= 0, & \mathbf{r} \in \mathcal{Q}_b; \quad u_b(\mathbf{r}) = u_f(\mathbf{r}), \\ \sigma_b \frac{\partial u_b(\mathbf{r})}{\partial n} &= \sigma_s \frac{\partial u_s(\mathbf{r})}{\partial n}, & \mathbf{r} \in \partial\mathcal{Q}_b, \end{aligned} \right\} \quad (2.6)$$

$$\left. \begin{aligned} \Delta u_s(\mathbf{r}) &= 0, & \mathbf{r} \in \mathcal{Q}_s; \quad u_s(\mathbf{r}) = u_b(\mathbf{r}), \\ \sigma_s \frac{\partial u_s(\mathbf{r})}{\partial n} &= 0, & \mathbf{r} \in \partial\mathcal{Q}_s. \end{aligned} \right\} \quad (2.7)$$

In the particular case that  $\mathbf{J}^p$  is a one dipole with moment  $\mathbf{Q}$ , i.e.

$$\mathbf{J}^p(\mathbf{r}) = \mathbf{Q}(\tau) \delta(\mathbf{r} - \tau), \quad \mathbf{r}, \tau \in \mathcal{Q}_c, \quad (2.8)$$

the solution of the first of equations (2.4) can be written in the form

$$\sigma_c u_c(\mathbf{r}, \tau) = \frac{1}{4\pi} \mathbf{Q}(\tau) \cdot \nabla_\tau \left[ \frac{1}{|\mathbf{r} - \tau|} + \tilde{u}_c(\mathbf{r}, \tau) \right], \quad \tau, \mathbf{r} \in \mathcal{Q}_c. \quad (2.9)$$

Indeed, by applying the operator  $\nabla_\tau$  to the equation

$$\Delta_\tau \left( -\frac{1}{4\pi} \frac{1}{|\mathbf{r} - \tau|} \right) = \delta(\mathbf{r} - \tau), \quad \mathbf{r}, \tau \in \mathcal{Q}_c,$$

it follows that a particular solution of the equation

$$\Delta_r \mathbf{F}(\mathbf{r}, \tau) = \nabla_r \delta(\mathbf{r} - \tau), \quad \mathbf{r}, \tau \in \Omega_c$$

is given by

$$-\frac{1}{4\pi} \nabla_r \frac{1}{|\mathbf{r} - \tau|}, \quad \mathbf{r}, \tau \in \Omega_c.$$

Hence,

$$\sigma_c u_c(\mathbf{r}, \tau) = -\frac{1}{4\pi} \mathbf{Q} \cdot \nabla_r \frac{1}{|\mathbf{r} - \tau|} + \tilde{u}(\mathbf{r}, \tau),$$

where  $\tilde{u}$  is a harmonic function. Writing  $\tilde{u}$  in the form of  $\mathbf{Q} \cdot \nabla_r \tilde{u} / 4\pi$  and noting that the action of  $\nabla_r$  on  $1/|\mathbf{r} - \tau|$  equals minus the action of  $\nabla_r$  on  $1/|\mathbf{r} - \tau|$ , equation (2.9) follows.

Introducing the notation

$$\begin{aligned} u_j(\mathbf{r}, \tau) &= \frac{1}{4\pi} \mathbf{Q}(\tau) \cdot \nabla_r \tilde{u}_j(\mathbf{r}, \tau), \quad j = f, b, s, \\ \mathbf{r} &\in \Omega_j, \quad \tau \in \Omega_c, \end{aligned} \quad (2.10)$$

equations (2.4)–(2.9) yield the following equations, where in all these equations  $\tau \in \Omega_c$ :

$$\left. \begin{aligned} \Delta \tilde{u}_c(\mathbf{r}, \tau) &= 0, \quad \mathbf{r} \in \Omega_c; \\ \frac{\partial}{\partial n} \left[ \frac{1}{|\mathbf{r} - \tau|} + \tilde{u}_c(\mathbf{r}, \tau) \right] &= \sigma_f \frac{\partial \tilde{u}_f(\mathbf{r}, \tau)}{\partial n}, \quad \mathbf{r} \in \partial \Omega_c, \end{aligned} \right\} \quad (2.11)$$

$$\left. \begin{aligned} \Delta \tilde{u}_f(\mathbf{r}, \tau) &= 0, \quad \mathbf{r} \in \Omega_f; \\ \tilde{u}_f(\mathbf{r}, \tau) &= \frac{1}{\sigma_c} \left[ \frac{1}{|\mathbf{r} - \tau|} + \tilde{u}_c(\mathbf{r}, \tau) \right], \quad \mathbf{r} \in \partial \Omega_c; \\ \sigma_f \frac{\partial \tilde{u}_f(\mathbf{r}, \tau)}{\partial n} &= \sigma_b \frac{\partial \tilde{u}_b(\mathbf{r}, \tau)}{\partial n}, \quad \mathbf{r} \in \partial \Omega_f, \end{aligned} \right\} \quad (2.12)$$

$$\left. \begin{aligned} \Delta \tilde{u}_b(\mathbf{r}, \tau) &= 0, \quad \mathbf{r} \in \Omega_b; \quad \tilde{u}_b(\mathbf{r}, \tau) = \tilde{u}_f(\mathbf{r}), \\ \sigma_b \frac{\partial \tilde{u}_b(\mathbf{r}, \tau)}{\partial n} &= \sigma_s \frac{\partial \tilde{u}_s(\mathbf{r}, \tau)}{\partial n}, \quad \mathbf{r} \in \partial \Omega_b, \end{aligned} \right\} \quad (2.13)$$

$$\left. \begin{aligned} \Delta \tilde{u}_s(\mathbf{r}, \tau) &= 0, \quad \mathbf{r} \in \Omega_s; \quad \tilde{u}_s(\mathbf{r}, \tau) = \tilde{u}_b(\mathbf{r}, \tau), \\ \mathbf{r} \in \partial \Omega_b; \quad \frac{\partial \tilde{u}_s(\mathbf{r}, \tau)}{\partial n} &= 0, \quad \mathbf{r} \in \partial \Omega_s. \end{aligned} \right\} \quad (2.14)$$

Equations (2.11)–(2.14) define well-posed boundary-value problems for the functions  $\tilde{u}_c$ ,  $\tilde{u}_f$ ,  $\tilde{u}_b$  and  $\tilde{u}_s$ . The solution of these problems depends only on the geometrical characteristics of the domains  $\{\Omega_c, \Omega_f, \Omega_b, \Omega_s\}$ , on the conductivities  $\{\sigma_c, \sigma_f, \sigma_b, \sigma_s\}$  and on  $\tau$ . Having obtained these functions, which are *independent* of  $\mathbf{Q}(\tau)$ , equations (2.9) and (2.10) yield  $u_c$ ,  $u_f$ ,  $u_b$ ,  $u_s$ .

An integral representation for  $\mathbf{B}$  can be derived by solving the second of equations (2.1), where

$$\mathbf{E}(\mathbf{r}) = -\nabla u_j(\mathbf{r}), \quad \mathbf{r} \in \Omega_j, \quad j = c, f, b, s.$$

Noting that

$$\nabla_r \cdot \left( -\frac{1}{4\pi} \nabla_r \frac{1}{|\mathbf{r} - \tau|} \right) = \delta(\mathbf{r} - \tau),$$

it follows that the expression in the above parentheses provides the fundamental solution of the operator  $\nabla_r$ .

Hence, the second of equations (2.1) yields

$$\begin{aligned} -\frac{4\pi}{\mu} \mathbf{B}(\mathbf{r}) &= \int_{\Omega} \mathbf{J}^p(\mathbf{r}') \times \nabla_r \frac{1}{|\mathbf{r} - \mathbf{r}'|} dV(\mathbf{r}') \\ &\quad - \sum_{j=1}^4 \int_{\Omega_j} \sigma_j \nabla u_j(\mathbf{r}') \times \nabla_r \frac{1}{|\mathbf{r} - \mathbf{r}'|} dV(\mathbf{r}'), \\ \mathbf{r} &\in \Omega_e, \end{aligned} \quad (2.15)$$

where subscripts  $j = 1, 2, 3, 4$  refer to  $c, f, b, s$ .

In the particular case that  $\mathbf{J}^p$  is the one dipole of equation (2.8), the first integral of the right-hand side of equation (2.15) becomes  $\mathbf{Q}(\tau) \times \nabla_r (1/(|\mathbf{r} - \tau|))$ . Furthermore, using Gauss theorem, it is possible to replace the volume integrals appearing in the second term of the right-hand side of equation (2.15) in terms of surface integrals. Hence, equation (2.15) yields

$$\begin{aligned} \frac{4\pi}{\mu} \mathbf{B}(\mathbf{r}, \tau) &= \mathbf{Q}(\tau) \times \nabla_r \frac{1}{|\mathbf{r} - \tau|} - \frac{1}{4\pi} \mathbf{Q}(\tau) \cdot \nabla_r \mathbf{H}(\mathbf{r}, \tau), \\ \mathbf{r} &\in \Omega_e, \quad \tau \in \Omega_c, \end{aligned} \quad (2.16)$$

where the function  $\mathbf{H}$ , which is *independent* of  $\mathbf{Q}(\tau)$ , is defined by the following equation:

$$\begin{aligned} \mathbf{H}(\mathbf{r}, \tau) &= \int_{\partial \Omega_c} \left\{ \left[ \frac{1}{|\mathbf{r}' - \tau|} + \tilde{u}_c(\mathbf{r}', \tau) \right] - \sigma_f \tilde{u}_f(\mathbf{r}', \tau) \right\} \hat{\mathbf{n}}(\mathbf{r}') \\ &\quad \times \nabla_{r'} \frac{1}{|\mathbf{r} - \mathbf{r}'|} dS(\mathbf{r}') \\ &\quad + \int_{\partial \Omega_f} [\sigma_f \tilde{u}_f(\mathbf{r}', \tau) - \sigma_b \tilde{u}_b(\mathbf{r}', \tau)] \hat{\mathbf{n}}(\mathbf{r}') \\ &\quad \times \nabla_{r'} \frac{1}{|\mathbf{r} - \mathbf{r}'|} dS(\mathbf{r}') \\ &\quad + \int_{\partial \Omega_b} [\sigma_b \tilde{u}_b(\mathbf{r}', \tau) - \sigma_s \tilde{u}_s(\mathbf{r}', \tau)] \hat{\mathbf{n}}(\mathbf{r}') \\ &\quad \times \nabla_{r'} \frac{1}{|\mathbf{r} - \mathbf{r}'|} dS(\mathbf{r}') \\ &\quad + \int_{\partial \Omega_s} \sigma_s \tilde{u}_s(\mathbf{r}', \tau) \hat{\mathbf{n}}(\mathbf{r}') \times \nabla_{r'} \frac{1}{|\mathbf{r} - \mathbf{r}'|} dS(\mathbf{r}'), \\ \mathbf{r} &\in \Omega_e, \quad \tau \in \Omega_c. \end{aligned} \quad (2.17)$$

In summary, the magnetic field  $\mathbf{B}(\mathbf{r}, \tau)$  computed outside the head and the electric potential  $u_s(\mathbf{r}, \tau)$  computed on the scalp corresponding to the one dipole (2.8) can be obtained as follows: first, obtain the harmonic functions  $\{\tilde{u}_c, \tilde{u}_f, \tilde{u}_b, \tilde{u}_s\}$  by solving the boundary-value problems defined in equations (2.11)–(2.14) and then determine the function  $\mathbf{H}$  using the definition (2.17); the magnetic field  $\{\mathbf{B}(\mathbf{r}, \tau), \mathbf{r} \in \Omega_e, \tau \in \Omega_c\}$  is given by equation (2.16) and the electric potential  $\{u_s(\mathbf{r}, \tau), \mathbf{r} \in \partial \Omega_s, \tau \in \Omega_c\}$  is given by equation (2.10) with  $j = s$ .

### 3. ARBITRARY GEOMETRY

After obtaining formulae for  $\mathbf{B}(\mathbf{r}, \tau)$  and  $u_s(\mathbf{r}, \tau)$  with *explicit*  $\mathbf{Q}(\tau)$  dependence, it is straightforward to compute  $\mathbf{B}(\mathbf{r})$  and  $u_s(\mathbf{r})$  in the case of a continuously

distributed current. This simply involves replacing  $\mathbf{Q}(\tau)$  by  $\mathbf{J}^P(\tau)$  and then integrating over  $dV(\tau)$ . By employing Green's theorem in the resulting expressions, it follows that EEG and MEG involve the divergence of  $\mathbf{J}^P$  and the curl of  $\mathbf{J}^P$ . This suggests the use of the 'Helmholtz decomposition' form for the current, which was actually already employed in Dassios *et al.* (2005) and (in a reduced form) in Fokas *et al.* (1996, 2004).

**Proposition 3.1.** *Consider the three-shell model specified by the domains  $\Omega_c$ ,  $\Omega_f$ ,  $\Omega_b$ ,  $\Omega_s$  modelling the spaces occupied by the cerebrum, the cerebrospinal fluid, the skull (bone) and the scalp, respectively. Let  $\sigma_c$ ,  $\sigma_f$ ,  $\sigma_b$ ,  $\sigma_s$  be the associated conductivities and let  $\mu$  be the permeability. Let  $\mathbf{J}^P(\tau)$  be a continuously distributed current with support in  $\Omega_c$ .*

Express  $\mathbf{J}^P(\tau)$  in the Helmholtz decomposition form

$$\mathbf{J}^P(\tau) = \nabla\Psi(\tau) + \nabla \times \mathbf{A}(\tau), \quad \nabla \cdot \mathbf{A}(\tau) = 0, \quad \tau \in \Omega_c. \quad (3.1)$$

The electric potential  $u_s(\mathbf{r})$ ,  $\mathbf{r} \in \partial\Omega_s$  is affected only by  $\Delta\Psi(\tau)$ , whereas the magnetic field  $\mathbf{B}(\mathbf{r})$ ,  $\mathbf{r} \in \Omega_e$ , is affected by  $\Delta\Psi(\tau)$  and by  $\Delta\mathbf{A}(\tau)$ . Furthermore, the radial part of  $\mathbf{B}(\mathbf{r})$ , i.e.  $\mathbf{r} \cdot \mathbf{B}(\mathbf{r})$ , is affected only by  $\Delta\Psi(\tau)$  and  $\Delta(\tau \cdot \mathbf{A}(\tau))$ .

In more detail, the following formulae are valid:

$$u_s(\mathbf{r}) = -\frac{1}{4\pi} \int_{\Omega_c} (\Delta\Psi(\tau)) \tilde{u}_s(\mathbf{r}, \tau) dV(\tau), \quad \mathbf{r} \in \partial\Omega_s, \quad (3.2)$$

$$\begin{aligned} \frac{4\pi}{\mu} \mathbf{B}(\mathbf{r}) = & -\int_{\Omega_c} (\Delta\mathbf{A}(\tau)) \frac{dV(\tau)}{|\mathbf{r} - \tau|} \\ & + \frac{1}{4\pi} \int_{\Omega_c} (\Delta\Psi(\tau)) \mathbf{H}(\mathbf{r}, \tau) dV(\tau), \quad \mathbf{r} \in \Omega_e, \end{aligned} \quad (3.3)$$

$$\begin{aligned} \frac{4\pi}{\mu} \mathbf{r} \cdot \mathbf{B}(\mathbf{r}) = & -\int_{\Omega_c} \Delta(\tau \cdot \mathbf{A}(\tau)) \frac{dV(\tau)}{|\mathbf{r} - \tau|} \\ & + \frac{1}{4\pi} \int_{\Omega_c} (\Delta\Psi(\tau)) \mathbf{r} \cdot \mathbf{H}(\mathbf{r}, \tau) dV(\tau), \quad \mathbf{r} \in \Omega_e. \end{aligned} \quad (3.4)$$

In equations (3.2)–(3.4),  $\tilde{u}_s$  and  $\mathbf{H}$  are defined in §2 in terms of the geometrical characteristics of  $\{\Omega_c, \Omega_f, \Omega_b, \Omega_s\}$ , the conductivities, and  $\tau$ .

*Proof.* Integrating equation (2.10) where  $j=s$  with respect to  $d\tau$  over  $\Omega_c$ , using Gauss theorem, and noting that

$$\nabla \cdot \mathbf{J}^P = \nabla \cdot (\nabla\Psi + \nabla \times \mathbf{A}) = \Delta\Psi,$$

equation (3.2) follows.

Similarly, integrating equation (2.16) with respect to  $d\tau$  over  $\Omega_c$ , using Gauss theorem to replace

$$\mathbf{J}^P \times \nabla_\tau \frac{1}{|\mathbf{r} - \tau|} \quad \text{with} \quad (\nabla_\tau \times \mathbf{J}^P) \frac{1}{|\mathbf{r} - \tau|}$$

and noting that

$$\begin{aligned} \nabla \times \mathbf{J}^P &= \nabla \times (\nabla\Psi + \nabla \times \mathbf{A}) \\ &= -\Delta\mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) = -\Delta\mathbf{A}, \end{aligned}$$

equation (3.3) follows.

For the derivation of equation (3.4), we will use the identity

$$\mathbf{r} \cdot \int_{\Omega_c} (\Delta\mathbf{A}(\tau)) \frac{dV(\tau)}{|\mathbf{r} - \tau|} = \int_{\Omega_c} \Delta(\tau \cdot \mathbf{A}(\tau)) \frac{dV(\tau)}{|\mathbf{r} - \tau|}, \quad (3.5)$$

which can be derived as follows. The left-hand side of equation (3.5) can be rewritten in the form

$$\begin{aligned} \mathbf{r} \cdot \int_{\Omega_c} (\Delta\mathbf{A}(\tau)) \frac{dV(\tau)}{|\mathbf{r} - \tau|} &= \int_{\Omega_c} (\Delta\mathbf{A}) \cdot \frac{(\mathbf{r} - \tau)}{|\mathbf{r} - \tau|} dV(\tau) \\ &+ \int_{\Omega_c} (\Delta\mathbf{A}) \cdot \tau \frac{dV(\tau)}{|\mathbf{r} - \tau|}. \end{aligned} \quad (3.6)$$

The first integral on the right-hand side of equation (3.6) equals

$$\begin{aligned} &\int_{\Omega_c} (\Delta\mathbf{A}) |\mathbf{r} - \tau|^2 \cdot \frac{(\mathbf{r} - \tau)}{|\mathbf{r} - \tau|^3} dV(\tau) \\ &= \int_{\Omega_c} (\Delta\mathbf{A}) |\mathbf{r} - \tau|^2 \cdot \nabla_\tau \frac{1}{|\mathbf{r} - \tau|} dV(\tau) \\ &= - \int_{\Omega_c} \nabla_\tau [(\Delta\mathbf{A}) |\mathbf{r} - \tau|^2] \frac{dV(\tau)}{|\mathbf{r} - \tau|} \\ &= -2 \int_{\Omega_c} (\tau - \mathbf{r}) \cdot \Delta\mathbf{A} \frac{dV(\tau)}{|\mathbf{r} - \tau|}. \end{aligned}$$

Replacing the first term of the right-hand side of equation (3.6) by the right-hand side of the above equation, equation (3.6) becomes equation (3.5) (multiplied by  $-1$ ).

The identity (3.5) implies that the first term of the right-hand side of equation (3.4) involves  $\tau \cdot \Delta\mathbf{A}$ , which equals  $\Delta(\tau \cdot \mathbf{A})$  owing to the identity

$$\tau \cdot \Delta\mathbf{A} = \Delta(\tau \cdot \mathbf{A}) + 2\nabla \cdot \mathbf{A}. \quad \blacksquare$$

**Remark 3.1.** The representation for  $\mathbf{B}(\mathbf{r})$ ,  $\mathbf{r} \in \Omega_e$  given by equation (3.3), implies that  $\nabla \times \mathbf{B}(\mathbf{r}) = 0$  for  $\mathbf{r} \in \Omega_e$ . Hence, there exists a function  $U(\mathbf{r})$ , the magnetic potential, such that

$$\mathbf{B}(\mathbf{r}) = \mu \nabla U(\mathbf{r}), \quad \mathbf{r} \in \Omega_e. \quad (3.7)$$

Equation (3.4) implies that the dot product  $\mathbf{r} \cdot \mathbf{B}$  involves only  $\Delta\Psi$  and  $\Delta(\tau \cdot \mathbf{A})$ . Hence,

$$\begin{aligned} \mathbf{r} \cdot \nabla U(\mathbf{r}) &\text{ depends only on } \Delta\Psi(\tau) \\ &\text{and on } \Delta(\tau \cdot \mathbf{A}(\tau)). \end{aligned}$$

The term  $\mathbf{r} \cdot \nabla U$  will give rise to a *first-order* partial differential equation (PDE); the solution  $U$  of this PDE (and hence  $\mathbf{B}$ ) will only depend on  $\Delta\Psi(\tau)$  and  $\Delta(\tau \cdot \mathbf{A}(\tau))$ . For this reason, in the remainder of this paper, emphasis will be placed on  $\mathbf{r} \cdot \mathbf{B}$  instead of  $\mathbf{B}$ .

In the particular cases of spherical and ellipsoidal geometries, the functions  $\{\tilde{u}_c, \tilde{u}_f, \tilde{u}_b, \tilde{u}_s\}$  can be expressed explicitly in terms of spherical and ellipsoidal harmonics, respectively (see Giapalaki & Kariotou 2006; Dassios & Fokas preprint a). Using these

representations, it is possible to obtain explicit representations for both  $\mathbf{B}$  and  $u_s$ . This will be done in §§4 and 5.

**Remark 3.2.** The current  $\mathbf{J}^p$  defined in equation (3.1), depends on the scalar function  $\Psi$  and on the three scalar functions  $\{A_j\}_1^3$  specifying the vector  $\mathbf{A}$ . However,  $\mathbf{A}$  satisfies the equation  $\nabla \cdot \mathbf{A} = 0$ , thus only two among the three functions  $\{A_j\}_1^3$  are independent. Equation (3.2) shows that  $u_s$  depends only on  $\Delta \Psi$ , whereas equation (3.4) shows that  $\mathbf{r} \cdot \mathbf{B}$  depends on  $\Delta \Psi$  and  $\Delta(\tau \cdot \mathbf{A})$ . Hence, EEG provides information about one of the three functions needed to specify  $\mathbf{J}^p$  (namely  $\Psi$ ), whereas MEG provides information about this function and one more function (namely the radial component of  $\mathbf{A}$ ).

#### 4. SPHERICAL GEOMETRY

Let  $\mathcal{Q}_c$  be a sphere of radius  $c_1$  and  $\mathcal{Q}_f$ ,  $\mathcal{Q}_b$  and  $\mathcal{Q}_s$  be concentric shells defined as follows:

$$\left. \begin{aligned} \mathcal{Q}_c : 0 \leq r < c_1, \\ \mathcal{Q}_f : c_1 < r < f_1, \\ \mathcal{Q}_b : f_1 < r < b_1, \\ \mathcal{Q}_s : b_1 < r < s_1. \end{aligned} \right\} \quad (4.1)$$

By making extensive use of the classical formula

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{n=0}^{\infty} \frac{\tau^n}{r^{n+1}} P_n(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}'), \quad r > \tau, \quad (4.2)$$

where  $P_n$  denotes the usual Legendre polynomials, it is shown in Dassios & Fokas (preprint a) that  $\tilde{u}_s$  is given by

$$\tilde{u}_s(\mathbf{r}, \tau) = \sum_{n=1}^{\infty} s_n \tau^n P_n(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}), \quad \tau < c_1, \quad r = s_1, \quad (4.3)$$

where the constant  $s_n$  is explicitly given in terms of  $\{c_1, f_1, b_1, s_1\}$  and  $\{\sigma_c, \sigma_f, \sigma_b, \sigma_s\}$  (see Dassios & Fokas preprint a).

For the spherical model, the dot product  $\mathbf{r} \cdot \mathbf{H}(\mathbf{r}, \tau)$  appearing in equation (3.4) vanishes. Indeed, for the integral along  $\partial \mathcal{Q}_j$ ,  $j = c, f, b, s$ ,

$$\mathbf{r} \cdot \hat{\mathbf{n}}(\mathbf{r}') \times \nabla_{r'} \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \mathbf{r} \cdot \frac{\mathbf{r}'}{r_j} \times \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} = 0,$$

where  $r_c = c_1$ ,  $r_f = f_1$ ,  $r_b = b_1$ ,  $r_s = s_1$ .

**Proposition 4.1.** Consider the three-shell spherical model specified by equations (4.1) and let the neuronal current  $\mathbf{J}^p(\tau)$ ,  $0 \leq \tau < c_1$ , be expressed in the form of equation (3.1). Then,

$$u_s(\mathbf{r}) = -\frac{1}{4\pi} \int_{\mathcal{Q}_c} \sum_{n=1}^{\infty} s_n (\Delta \Psi(\tau)) \tau^n P_n(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}) dV(\tau), \quad r = s_1 \quad (4.4)$$

and

$$\frac{4\pi}{\mu} \mathbf{r} \cdot \mathbf{B}(\mathbf{r}) = - \int_{\mathcal{Q}_c} \sum_{n=1}^{\infty} \Delta(\tau A^r(\tau)) \frac{\tau^n}{r^{n+1}} P_n(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}) dV(\tau), \quad r > s_1, \quad (4.5)$$

where  $A^r(\tau)$  denotes the radial component of  $\mathbf{A}(\tau)$  and the constant  $s_n$  is explicitly given in terms of the conductivities and of  $(c_1, f_1, b_1, s_1)$  (see Dassios & Fokas preprint a).

The expressions in equations (4.4) and (4.5) can be simplified into the following expressions:

$$u_s(\mathbf{r}) = - \sum_{n=1}^{\infty} \sum_{m=-n}^n \frac{s_n}{2n+1} c_1^{n+1} [c_1 \dot{\psi}_n^m(c_1) - n \psi_n^m(c_1)] \times Y_n^m(\hat{\mathbf{r}}), \quad r = s_1 \quad (4.6)$$

and

$$\frac{4\pi}{\mu} \mathbf{r} \cdot \mathbf{B}(\mathbf{r}) = - \sum_{n=1}^{\infty} \sum_{m=-n}^n \frac{c_1^{n+2}}{2n+1} [c_1 \dot{a}_n^m(c_1) - (n-1) a_n^m(c_1)] \frac{Y_n^m(\hat{\mathbf{r}})}{r^{n+1}}, \quad r > s_1, \quad (4.7)$$

where  $\dot{\psi}_n^m$ ,  $\dot{a}_n^m$  denote derivatives with respect to  $\tau$ , and  $\psi_n^m(\tau)$  and  $a_n^m(\tau)$  are the  $\tau$ -dependent parts of the expansions of  $\Psi(\tau)$  and  $A^r(\tau)$  in spherical harmonics, i.e.

$$\Psi(\tau) = \sum_{n=1}^{\infty} \sum_{m=-n}^n \psi_n^m(\tau) Y_n^m(\hat{\mathbf{r}}) \quad (4.8)$$

and

$$A^r(\tau) = \sum_{n=1}^{\infty} \sum_{m=-n}^n a_n^m(\tau) Y_n^m(\hat{\mathbf{r}}), \quad 0 < \tau < c_1. \quad (4.9)$$

*Proof.* Replacing in equations (3.2) and (3.4),  $\tilde{u}_s(\mathbf{r}, \tau)$  and  $1/|\mathbf{r} - \mathbf{r}'|$  by the right-hand side of equations (4.3) and (4.2), and noting that  $\mathbf{r} \cdot \mathbf{H} = 0$ , equations (3.2) and (3.4) become equations (4.4) and (4.5).

Recall that

$$dV(\tau) = \tau^2 \sin \theta d\tau d\theta d\varphi. \quad (4.10)$$

Replacing in equation (4.4),  $\Psi$  by the right-hand side of equation (4.8) and using the identity

$$\begin{aligned} \tau^2 \Delta \psi_n^m(\tau) Y_n^m(\hat{\mathbf{r}}) &= \left[ \frac{d}{d\tau} \left( \tau^2 \frac{d}{d\tau} \psi_n^m(\tau) \right) \right. \\ &\quad \left. - n(n+1) \psi_n^m(\tau) \right] Y_n^m(\hat{\mathbf{r}}) \end{aligned} \quad (4.11)$$

and the orthogonality condition

$$\int_0^{2\pi} d\varphi \int_0^\pi d\theta Y_n^m(\hat{\mathbf{r}}) P_{n'}(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') \sin \theta = \frac{4\pi}{2n+1} Y_n^m(\hat{\mathbf{r}}) \delta_{nn'}, \quad (4.12)$$

equation (4.4) yields

$$u_s(\mathbf{r}) = - \sum_{n=1}^{\infty} \sum_{m=-n}^n \frac{s_n}{2n+1} \left\{ \int_0^{c_1} d\tau \left[ \frac{d}{d\tau} \left( \tau^2 \frac{d}{d\tau} \psi_n^m(\tau) \right) \right. \right. \\ \left. \left. - n(n+1) \psi_n^m(\tau) \right] \tau^n \right\} Y_n^m(\hat{\mathbf{r}}). \quad (4.13)$$

Integration by parts implies

$$\begin{aligned} &\int_0^{c_1} d\tau \left[ \frac{d}{d\tau} \left( \tau^2 \frac{d}{d\tau} \psi_n^m(\tau) \right) \right] \tau^n \\ &= c_1^{n+1} [c_1 \dot{\psi}_n^m(c_1) - n \psi_n^m(c_1)] + n(n+1) \\ &\quad \times \int_0^{c_1} d\tau \psi_n^m(\tau) \tau^n, \end{aligned}$$

hence equation (4.13) becomes equation (4.6).

The derivation of equation (4.7) is similar to that of equation (4.6), where  $\psi_n^m$  is now replaced by  $\tau a_n^m$ . ■

**Remark 4.1.** Equations (4.6) and (4.7) show that the electrical potential  $u_s(\mathbf{r})$  evaluated on the sphere  $r=s_1$  and the radial component of the magnetic field  $\mathbf{r} \cdot \mathbf{B}$  evaluated in the exterior of the head,  $r>s_1$ , depend respectively only on the following expressions:

$$\left. \begin{array}{l} c_1 \dot{\psi}_n^m(c_1) - n \psi_n^m(c_1) \\ c_1 \dot{a}_n^m(c_1) - (n-1) a_n^m(c_1), \end{array} \right\} \quad (4.14)$$

where  $\psi_n^m(\tau)$  and  $a_n^m(\tau)$  are the  $\tau$ -dependent parts of the expansions of  $\Psi(\tau)$  and  $A^\tau(\tau)$  in terms of spherical harmonics (see equations (4.8) and (4.9)). The function  $A^\tau(\tau)$  is the radial component of the vector function  $\mathbf{A}(\tau)$ . Taking into consideration that  $\mathbf{A}$  satisfies  $\nabla \cdot \mathbf{A}=0$ , it follows that one of the two independent functions specifying  $\mathbf{A}$  remains arbitrary. In addition, the knowledge of the expressions appearing in (4.14) is insufficient for determining  $\psi_n^m(\tau)$  and  $a_n^m(\tau)$ . Regarding the latter problem, the situation is similar to that occurring in the inverse gravimetric problem (see the review by Michel & Fokas (2008)); for the gravimetric problem, it is often assumed that the associated radial function is either harmonic or bi-harmonic. Assuming that  $A^\tau(\tau)$  is harmonic, i.e.  $a_n^m(\tau) = \lambda_n^m \tau^n$ , the second of the expressions appearing in (4.14) yields  $\lambda_n^m c_1^n$  and hence the coefficients  $\lambda_n^m$  can be determined from the MEG measurements. Furthermore, noting that the equation  $\nabla \cdot \mathbf{A}=0$  takes the form

$$\sin \theta \frac{\partial}{\partial \tau} (\tau^2 A^\tau) + \tau \frac{\partial}{\partial \theta} (\sin \theta A^\theta) + \tau \frac{\partial}{\partial \varphi} (A^\varphi) = 0 \quad (4.15)$$

and employing the expansions

$$\frac{\partial}{\partial \theta} (\sin \theta A^\theta) = \sin \theta \sum_{n=1}^{\infty} \sum_{m=-n}^n b_n^m(\tau) Y_n^m(\hat{\tau}), \quad (4.16)$$

$$\frac{\partial}{\partial \varphi} (A^\varphi) = \sin \theta \sum_{n=1}^{\infty} \sum_{m=-n}^n c_n^m(\tau) Y_n^m(\hat{\tau}), \quad (4.17)$$

for the unknown functions  $A^\theta$  and  $A^\varphi$ , equation (4.15) yields

$$\frac{d}{d\tau} (\tau^2 a_n^m(\tau)) + \tau b_n^m(\tau) + \tau c_n^m(\tau) = 0, \quad 0 < \tau < c_1. \quad (4.18)$$

The harmonicity assumption for  $a_n^m(\tau)$  implies the following relation for the unknown functions  $b_n^m(\tau)$  and  $c_n^m(\tau)$ :

$$\tau(b_n^m(\tau) + c_n^m(\tau)) = -(n+2)\lambda_n^m \tau^{n+1}. \quad (4.19)$$

The harmonicity assumption is inappropriate for  $\Psi(\tau)$ , since in this case the first of the expressions appearing in (4.14) vanishes. On the other hand, assuming that  $\Psi(\tau)$  is *bi-harmonic*, i.e.  $\psi_n^m(\tau) = k_n^m \tau^{n+2}$ , the first of the expressions in (4.14) yields  $2k_n^m c_1^{n+2}$  and hence the coefficients  $k_n^m$  can be determined from the EEG measurements.

**Remark 4.2.** Defining the magnetic potential  $U(\mathbf{r})$  by equation (3.7), using the identity

$$\frac{1}{\mu} \mathbf{r} \cdot \mathbf{B}(\mathbf{r}) = \mathbf{r} \cdot \nabla U(\mathbf{r}) = \mathbf{r} \frac{\partial U}{\partial r}$$

and integrating equations (4.5) and (4.7) with respect to  $r$  from  $r$  to  $\infty$  (where  $U$  vanishes), equations (4.5) and (4.7) yield the following expressions for the magnetic potential  $U$ :

$$U(\mathbf{r}) = \frac{1}{4\pi} \int_{\Omega_c} \sum_{n=1}^{\infty} \frac{1}{(n+1)r} \Delta(\tau A^\tau(\tau)) \left( \frac{\tau}{r} \right)^n \times P_n(\hat{\tau} \cdot \hat{\mathbf{r}}) dV(\tau), \quad r > s_1 \quad (4.20)$$

and

$$U(\mathbf{r}) = \frac{1}{4\pi} \sum_{n=1}^{\infty} \sum_{m=-n}^n \frac{c_1^{n+2}}{(n+1)(2n+1)} [c_1 \dot{a}_n^m(c_1) - (n-1) a_n^m(c_1)] \frac{Y_n^m(\hat{\tau})}{r^{n+1}}, \quad r > s_1. \quad (4.21)$$

## 5. ELLIPSOIDAL GEOMETRY

The surfaces  $\partial\Omega_j$ ,  $j=c, f, b, s$  are now confocal ellipsoidal surfaces with the following characteristics:

$$\left. \begin{array}{l} \partial\Omega_c : 0 < c_3 < c_2 < c_1, \quad \partial\Omega_f : 0 < f_3 < f_2 < f_1, \\ \partial\Omega_b : 0 < b_3 < b_2 < b_1, \quad \partial\Omega_s : 0 < s_3 < s_2 < s_1. \end{array} \right\} \quad (5.1)$$

This means that the surface  $\partial\Omega_c$  is defined by the equation

$$\frac{x_1^2}{c_1^2} + \frac{x_2^2}{c_2^2} + \frac{x_3^2}{c_3^2} = 1 \quad (5.2)$$

and similarly for the other surfaces.

The above surfaces have the same semi-focal distances  $\{h_j\}_{j=1}^3$ , where

$$\left. \begin{array}{l} h_1^2 = s_2^2 - s_3^2 = b_2^2 - b_3^2 = f_2^2 - f_3^2 = c_2^2 - c_3^2, \\ h_2^2 = s_1^2 - s_3^2 = b_1^2 - b_3^2 = f_1^2 - f_3^2 = c_1^2 - c_3^2, \\ h_3^2 = s_1^2 - s_2^2 = b_1^2 - b_2^2 = f_1^2 - f_2^2 = c_1^2 - c_2^2. \end{array} \right\} \quad (5.3)$$

The Cartesian coordinates  $(\tau_1, \tau_2, \tau_3)$  of a point  $\tau$  are related with their ellipsoidal coordinates  $(\rho, \mu, \nu)$  by the following equations:

$$\left. \begin{array}{l} \tau_1 = \frac{\rho \mu \nu}{h_2 h_3}, \\ \tau_2 = \frac{1}{h_1 h_3} \sqrt{\rho^2 - h_3^2} \sqrt{\mu^2 - h_3^2} \sqrt{h_3^2 - \nu^2}, \\ \tau_3 = \frac{1}{h_2 h_1} \sqrt{\rho^2 - h_2^2} \sqrt{h_2^2 - \mu^2} \sqrt{h_2^2 - \nu^2}, \\ h_2 \leq \rho < \infty, \quad h_2 \leq \mu \leq h_3, \quad -h_3 \leq \nu < h_3. \end{array} \right\} \quad (5.4)$$

The analogue of equation (4.2) is now the classical formula

$$\frac{1}{|\mathbf{r} - \tau|} = \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \frac{4\pi}{(2n+1)\gamma_n^m} \mathbb{E}_n^m(\tau) \mathbb{F}_n^m(\mathbf{r}), \quad \rho > R, \quad (5.5)$$

where

$$\left. \begin{aligned} \mathbb{E}_n^m(\tau) &= E_n^m(\rho)E_n^m(\mu)E_n^m(\nu), \\ \mathbb{F}_n^m(\mathbf{r}) &= (2n+1)\mathbb{E}_n^m(\mathbf{r})I_n^m(R), \\ I_n^m(R) &= \int_R^\infty \frac{d\rho}{(E_n^m(\rho))^2 \sqrt{\rho^2 - h_3^2} \sqrt{\rho^2 - h_2^2}} \end{aligned} \right\} \quad (5.6)$$

and  $E_n^m(\rho)$  and  $\gamma_n^m$  are defined as follows:  $E_n^m(\rho)$ , the interior ellipsoidal harmonics, satisfy the Lamé equation

$$\left. \begin{aligned} L_n^m E_n^m(\rho) &= 0, \\ L_n^m &= \frac{d}{d\rho} \sqrt{\rho^2 - h_3^2} \sqrt{\rho^2 - h_2^2} \frac{d}{d\rho} \\ &+ \frac{(h_2^2 + h_3^2)P_n^m - n(n+1)\rho^2}{\sqrt{\rho^2 - h_3^2} \sqrt{\rho^2 - h_2^2}}, \end{aligned} \right\} \quad (5.7)$$

$P_n^m$  constant; the constant  $\gamma_n^m$  is defined by

$$\left. \begin{aligned} \gamma_n^m &= \int_{h_3}^{h_2} \int_{-h_3}^{h_3} (E_n^m(\mu))^2 (E_n^m(\nu))^2 dS_e, \\ dS_e &= \frac{(\mu^2 - \nu^2) d\mu d\nu}{\sqrt{\mu^2 - h_3^2} \sqrt{h_2^2 - \mu^2} \sqrt{h_3^2 - \nu^2} \sqrt{h_2^2 - \nu^2}}. \end{aligned} \right\} \quad (5.8)$$

The formulae for  $\tilde{u}_j(\mathbf{r}, \tau)$  are given in Giapalaki & Kariotou (2006). Using these important formulae, it follows that  $\tilde{u}_j$ ,  $j=b, f, s$  evaluated on the surface  $\partial\Omega_l$ ,  $l=c, b, f, s$  are given by

$$\tilde{u}_j(\mathbf{r}', \tau) = 4\pi \sum_{n=1}^{\infty} \sum_{m=1}^{2n+1} \frac{1}{\gamma_n^m} C_{jn}^{lm} \mathbb{E}_n^m(\tau) E_n^m(M) E_n^m(N),$$

$$\mathbf{r}' \in \partial\Omega_l, \tau \in \Omega_c, j = b, f, s, l = c, b, f, s, \quad (5.9)$$

where the constants  $C_{jn}^{lm}$  depend on the conductivities and on the geometrical constants appearing in equation (5.1) (see equations (A 1)–(A 5) of appendix A). For example, for  $\tilde{u}_s$  evaluated on  $\partial\Omega_s$ , the associated constant  $C_{sn}^{sm}$  is given by equation (A 1). The function

$$\frac{1}{\sigma_c} \left[ \frac{1}{|\mathbf{r}' - \tau|} + \tilde{u}_c(\mathbf{r}', \tau) \right], \quad \mathbf{r}' \in \partial\Omega_c, \tau \in \Omega_c$$

is also given by the right-hand side of equation (5.9) with the relevant constant equal to  $C_{cm}^{cn}$ .

In contrast to the case of spherical geometry, the dot product of  $\mathbf{r}$  with  $\mathbf{H}$  does not vanish in the ellipsoidal case. Thus, it is now necessary to evaluate the surface integrals appearing in the definition of  $\mathbf{H}$  (see equation (2.17)). It is shown in appendix A that

$$\frac{1}{4\pi} \mathbf{H}(\mathbf{r}, \tau) = \sum_{n=1}^{\infty} \sum_{m=1}^{2n+1} \frac{\mathbb{F}_n^m(\mathbf{r})}{(2n+1)\gamma_n^m} \sum_{l=1, l \neq m}^{2n+1} \mathbf{H}_n^{ml} \mathbb{E}_n^l(\tau), \quad (5.10)$$

where the Cartesian components of the constant vector  $\mathbf{H}_n^{ml}$  depend on the conductivities and on the geometrical constants appearing in equation (5.1).

Using equation (5.10) as well as the classical formula (5.5), equation (2.16) becomes

$$\frac{\mathbf{B}(\mathbf{r}, \tau)}{\mu} = \sum_{n=1}^{\infty} \sum_{m=1}^{2n+1} \frac{\mathbb{F}_n^m(\mathbf{r})}{(2n+1)\gamma_n^m} \left\{ \mathbf{Q}(\tau) \times \nabla_{\tau} \mathbb{E}_n^m(\tau) \right. \\ \left. - \mathbf{Q}(\tau) \cdot \nabla_{\tau} \sum_{l=1}^{2n+1} \frac{1}{4\pi} \mathbf{H}_n^{ml} \mathbb{E}_n^l(\tau) \right\}, \quad (5.11)$$

where  $\mathbf{r} \in \Omega_e$  and  $\tau \in \Omega_c$ .

**Proposition 5.1.** Consider the three-shell ellipsoidal model specified by equations (5.1) and let the neuronal current  $\mathbf{J}^p(\tau)$ ,  $\tau \in \Omega_c$ , be expressed in the form (3.1). Then,

$$\left. \begin{aligned} \mathbf{u}_s(\mathbf{r}) &= -\frac{1}{4\pi} \int_{\Omega_c} \sum_{n=1}^{\infty} \sum_{m=1}^{2n+1} \frac{1}{\gamma_n^m} C_{sn}^{sm} (\Delta \Psi(\tau)) \\ &\times \mathbb{E}_n^m(\tau) E_n^m(M) E_n^m(N) dV(\tau), \quad \mathbf{r} \in \partial\Omega_s \end{aligned} \right\} \quad (5.12)$$

and

$$\left. \begin{aligned} \frac{\mathbf{r} \cdot \mathbf{B}(\mathbf{r})}{\mu} &= - \int_{\Omega_c} \sum_{n=1}^{\infty} \sum_{m=1}^{2n+1} \frac{1}{(2n+1)\gamma_n^m} \left\{ \Delta(\tau \cdot \mathbf{A}(\tau)) \mathbb{E}_n^m(\tau) \right. \\ &\left. - (\Delta \Psi(\tau)) \sum_{l=1, l \neq m}^{2n+1} \frac{1}{4\pi} \mathbf{r} \cdot \mathbf{H}_n^{ml} \mathbb{E}_n^l(\tau) \right\} \mathbb{F}_n^m(\mathbf{r}) dV(\tau), \\ \mathbf{r} &\in \Omega_e. \end{aligned} \right\} \quad (5.13)$$

In equation (5.12), the ellipsoidal coordinates of the point  $\mathbf{r}$  on the surface  $\partial\Omega_s$  are  $(s_1, M, N)$ ; the constants  $C_{sn}^{sm}$  as well as the Cartesian coordinates of the constant vector  $\mathbf{H}_n^{ml}$  can be explicitly computed in terms of the conductivities and of the constants  $(c_j, f_j, b_j, s_j)$ ,  $j=1, 2, 3$ .

The expressions in equations (5.12) and (5.13) can be simplified into the following expressions:

$$\left. \begin{aligned} \mathbf{u}_s(\mathbf{r}) &= -\frac{1}{4\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{2n+1} C_{sn}^{sm} \sqrt{c_1^2 - h_3^2} \sqrt{c_1^2 - h_2^2} [\psi_n^m(c_1) E_n^m(c_1) \\ &- \psi_n^m(c_1) \dot{E}_n^m(c_1)] E_n^m(M) E_n^m(N), \quad \mathbf{r} \in \partial\Omega_s \end{aligned} \right\} \quad (5.14)$$

and

$$\left. \begin{aligned} \frac{\mathbf{r} \cdot \mathbf{B}(\mathbf{r})}{\mu} &= - \sum_{n=1}^{\infty} \sum_{m=1}^{2n+1} \frac{\sqrt{c_1^2 - h_3^2} \sqrt{c_1^2 - h_2^2}}{2n+1} \\ &\times \left\{ [\dot{a}_n^m(c_1) E_n^m(c_1) - a_n^m(c_1) \dot{E}_n^m(c_1)] \right. \\ &\left. - \sum_{l=1, l \neq m}^{2n+1} \frac{1}{4\pi} \mathbf{r} \cdot \mathbf{H}_n^{ml} [\dot{\psi}_n^l(c_1) E_n^l(c_1) \right. \\ &\left. - \psi_n^l(c_1) \dot{E}_n^l(c_1)] \right\} \mathbb{F}_n^m(\mathbf{r}), \quad \mathbf{r} \in \Omega_e, \end{aligned} \right\} \quad (5.15)$$

where  $\psi_n^m(\tau)$  and  $a_n^m(\tau)$  are the radial parts of the expansions of  $\Psi(\tau)$  and of  $\tau \cdot \mathbf{A}(\tau)$  in ellipsoidal harmonics, i.e.

$$\Psi(\tau) = \sum_{n=1}^{\infty} \sum_{m=1}^{2n+1} \psi_n^m(\rho) E_n^m(\mu) E_n^m(\nu) \quad (5.16)$$

and

$$\tau \cdot \mathbf{A}(\tau) = \sum_{n=1}^{\infty} \sum_{m=1}^{2n+1} a_n^m(\rho) E_n^m(\mu) E_n^m(\nu), \quad \tau \in \Omega_c. \quad (5.17)$$

*Proof.* Replacing in equation (3.2), the function  $\tilde{u}_s(\mathbf{r}, \tau)$  by the right-hand side of equation (5.9) with  $j=l=s$ ,

equation (3.2) becomes equation (5.12). Similarly, replacing in equation (3.4)  $1/|\mathbf{r}-\mathbf{r}|$  and  $\mathbf{H}/4\pi$  by the right-hand side of equations (5.5) and (5.10), equation (3.4) becomes equation (5.13).

The volume differential for  $\mathcal{Q}_c$  is given by

$$dV(\tau) = h_\rho h_\mu h_\nu d\rho d\mu d\nu, \quad (5.18)$$

where the functions  $h_\rho$ ,  $h_\mu$  and  $h_\nu$  are defined by

$$\left. \begin{aligned} h_\rho &= \frac{\sqrt{\rho^2 - \mu^2}}{\sqrt{\rho^2 - h_3^2}} \frac{\sqrt{\rho^2 - \nu^2}}{\sqrt{\rho^2 - h_2^2}}, \\ h_\mu &= \frac{\sqrt{\mu^2 - \nu^2}}{\sqrt{h_2^2 - \mu^2}} \frac{\sqrt{\rho^2 - \mu^2}}{\sqrt{\mu^2 - h_3^2}}, \\ h_\nu &= \frac{\sqrt{\mu^2 - \nu^2}}{\sqrt{h_2^2 - \nu^2}} \frac{\sqrt{\rho^2 - \nu^2}}{\sqrt{h_3^2 - \nu^2}}. \end{aligned} \right\} \quad (5.19)$$

The definitions of  $dS_e$  (see equation (5.8)),  $h_\mu$  and  $h_\nu$  imply that

$$dS_e = \frac{h_\mu h_\nu}{\sqrt{\rho^2 - \mu^2} \sqrt{\rho^2 - \nu^2}} d\mu d\nu,$$

hence

$$dV(\tau) = \frac{(\rho^2 - \mu^2)(\rho^2 - \nu^2)}{\sqrt{\rho^2 - h_3^2} \sqrt{\rho^2 - h_2^2}} d\rho dS_e. \quad (5.20)$$

The analogue of equation (4.12) is the following orthogonality condition

$$\begin{aligned} &\int_{h_2}^{h_3} \int_{-h_3}^{h_3} E_n^m(\mu) E_n^m(\nu) E_{n'}^{m'}(\mu) E_{n'}^{m'}(\nu) dS_e \\ &= \gamma_n^m \delta_{nm} \delta_{mm'}. \end{aligned} \quad (5.21)$$

Furthermore, the analogue of equation (4.11) is the following identity (Dassios & Fokas preprint c):

$$\begin{aligned} &\frac{(\rho^2 - \mu^2)(\rho^2 - \nu^2)}{\sqrt{\rho^2 - h_3^2} \sqrt{\rho^2 - h_2^2}} \Delta \psi_n^m(\rho) E_n^m(\mu) E_n^m(\nu) \\ &= (L_n^m \psi_n^m(\rho)) E_n^m(\mu) E_n^m(\nu), \end{aligned} \quad (5.22)$$

where the linear operator  $L_n^m$  is defined in equation (5.7).

Replacing in equation (5.12)  $\Psi(\tau)$  and  $dV(\tau)$  by the right-hand side of equations (5.16) and (5.20), and then making use of equations (5.22) and (5.21), equation (5.12) becomes

$$\begin{aligned} u_s(\mathbf{r}) &= -\frac{1}{4\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{2n+1} C_{sn}^{sm} E_n^m(M) E_n^m(N) \\ &\times \int_{h_2}^{c_1} (L_n^m \psi_n^m(\rho)) E_n^m(\rho) d\rho, \quad \mathbf{r} \in \partial \mathcal{Q}_s. \end{aligned} \quad (5.23)$$

Integration by parts implies

$$\begin{aligned} &\int_{h_2}^{c_1} E_n^m(\rho) \frac{d}{d\rho} \left[ \sqrt{\rho^2 - h_3^2} \sqrt{\rho^2 - h_2^2} \frac{d}{d\rho} \psi_n^m(\rho) \right] d\rho \\ &= \int_{h_2}^{c_1} \psi_n^m(\rho) \frac{d}{d\rho} \left[ \sqrt{\rho^2 - h_3^2} \sqrt{\rho^2 - h_2^2} \frac{d}{d\rho} E_n^m(\rho) \right] d\rho \\ &+ \sqrt{c_1^2 - h_3^2} \\ &\times \sqrt{c_1^2 - h_2^2} [\dot{\psi}_n^m(c_1) E_n^m(c_1) - \psi_n^m(c_1) \dot{E}_n^m(c_1)]. \end{aligned}$$

Hence, equation (5.23) becomes equation (5.14). ■

The derivative of equation (5.15) is similar. ■

## 6. CONCLUSIONS

The three-shell model of electro-magneto-encephalography has been analysed in arbitrary, spherical and ellipsoidal geometries. By using the Helmholtz decomposition for the current, i.e. by expressing  $\mathbf{J}^p$  in terms of  $\Psi$  and  $\mathbf{A}$ , where  $\mathbf{A}$  satisfies the equation  $\nabla \cdot \mathbf{A} = 0$  (see equation (3.1)), it has been shown that in general the electric potential evaluated on the scalp involves only  $\Delta \Psi$  (see equation (3.2)), whereas the radial component of the magnetic field evaluated outside the head involves only  $\Delta \Psi$  and  $\Delta(\tau \cdot \mathbf{A})$  (see equation (3.4)).

In the particular case of spherical geometry, the relevant formulae simplify:  $u_s$  and  $\mathbf{r} \cdot \mathbf{B}$  depend, respectively, on

$$c_1 \dot{\psi}_n^m(c_1) - n \psi_n^m(c_1) \quad \text{and} \quad c_1 \dot{a}_n^m(c_1) - (n-1) a_n^m(c_1),$$

where  $\psi_n^m$  and  $a_n^m$  are the  $\tau$ -dependent parts of the expansions of  $\Psi(\tau)$  and  $A^{\tau}$  in terms of spherical harmonics (see equations (4.8) and (4.9)).

Similarly, in the case of ellipsoidal geometry,  $u_s$  and  $\mathbf{r} \cdot \mathbf{B}$  depend, respectively, on

$$\dot{\psi}_n^m(c_1) E_n^m(c_1) - \psi_n^m(c_1) \dot{E}_n^m(c_1)$$

and

$$\dot{a}_n^m(c_1) E_n^m(c_1) - a_n^m(c_1) \dot{E}_n^m(c_1)$$

where  $\psi_n^m(\rho)$  and  $a_n^m(\rho)$  are the  $\rho$ -dependent parts of the expansions of  $\Psi(\tau)$  and  $\tau \cdot \mathbf{A}$  in terms of ellipsoidal harmonics (see equations (5.16) and (5.17)).

The questions of determining completely the functions  $\psi_n^m$  and  $a_n^m$ , as well as determining the tangential part of  $\mathbf{A}$ , remain open. A possible approach to this non-uniqueness question is to use the minimization of the  $L_2$ -norm of  $\mathbf{J}^p$ . In this respect, we recall that for the inverse gravimetric problem (which has certain similarities with the inverse MEG problem), the mass density that minimizes the  $L_2$ -norm is the harmonic density (Michel & Fokas 2008). A similar result for the MEG problem was obtained in Fokas *et al.* (1996, 2004). However, the representation of  $U(\tau)$  used in Fokas *et al.* (1996, 2004) is *different* from the one used here. This is due to the fact that the representation for the current  $\mathbf{J}^p$  used here (see equation (3.1)) is different from the one used in Fokas *et al.* (1996, 2004). The latter representation is quite convenient for the case that one considers *only* MEG, but it is *inappropriate* for the case that MEG and EEG are used *simultaneously*. Indeed, in the representation of Fokas *et al.* (1996, 2004), the radial component of  $\mathbf{J}^p$  is arbitrary and then the contributions of this part to MEG and EEG do *not* uncouple.

The unique current corresponding to the minimal  $L_2$ -norm assumption, and the numerical implementation of the relevant inverse algorithm using real data from the Brain Unit of MRC, Cambridge, will be presented in another publication.

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## APPENDIX A

Explicit representations for the functions  $\tilde{u}_j(\mathbf{r}, \tau)$ ,  $j=c, b, f, c$  for  $\mathbf{r} \in \Omega_j$  can be deduced from the representations (5.1)–(5.5) of Giapalaki & Kariotou (2006). Indeed, these equations immediately imply the following formulae for the constants  $C_{jn}^{lm}$ :

$$C_{sn}^{sm} = \frac{1}{\dot{E}_n^m(s_1) G_{3,n}^m}, \quad (\text{A } 1)$$

$$C_{sm}^{bn} = C_{bm}^{bn} = C_{sn}^{sm} + \frac{E_n^m(b_1)}{G_{3,n}^m} [I_n^m(b_1) - I_n^m(s_1)], \quad (\text{A } 2)$$

$$C_{bm}^{fn} = C_{fm}^{fn} = C_{sm}^{bn} + \frac{E_n^m(f_1)}{\sigma_b} [I_n^m(f_1) - I_n^m(b_1)] \frac{G_{1,n}^m}{G_{3,n}^m}, \quad (\text{A } 3)$$

$$C_{fm}^{cn} = C_{bm}^{fn} + \frac{E_n^m(c_1)}{\sigma_f} [I_n^m(c_1) - I_n^m(f_1)] \frac{G_{2,n}^m}{G_{3,n}^m}, \quad (\text{A } 4)$$

$$C_{cm}^{cn} = C_{fm}^{cn} + \frac{E_n^m(b_1)}{G_{3,n}^m} [I_n^m(b_1) - I_n^m(s_1)], \quad (\text{A } 5)$$

where the constant  $G_{j,n}^m$ ,  $j=1, 2, 3$ , is given in equations (5.6) and (5.7) of Giapalaki & Kariotou (2006).

In order to derive equation (5.10), we first note that the unit normal  $\hat{\mathbf{n}}$  to the ellipsoidal surface  $\partial\Omega_c$  coincides with the evaluation of the unit vector  $\hat{\mathbf{p}}$  (associated with the ellipsoidal coordinates  $\rho, \mu, \nu$ ) on the surface  $\partial\Omega_c$ , which is given by (Dassios & Kariotou 2003; Dassios *et al.* 2007b)

$$\hat{\mathbf{n}}(\mathbf{r}') = \frac{c_1 c_2 c_3}{\sqrt{c_1^2 - \mu^2} \sqrt{c_1^2 - \nu^2}} \sum_{j=1}^3 \frac{x'_j \hat{\mathbf{x}}_j}{c_j}, \quad \mathbf{r}' \in \partial\Omega_c, \quad (\text{A } 6)$$

where  $(x'_1, x'_2, x'_3)$  are the Cartesian coordinates of the point  $\mathbf{r}'$  and  $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3)$  denote the unit vectors  $(\mathbf{i}, \mathbf{j}, \mathbf{k})$  along the Cartesian axes.

The surface differential  $dS(\mathbf{r}')$  of the surface  $\partial\Omega_c$  equals  $h_\mu(\mathbf{r}') h_\nu(\mathbf{r}') d\mu d\nu$ ,  $\mathbf{r}' \in \partial\Omega_c$ , where  $h_\mu$  and  $h_\nu$  are defined in equation (5.19). By evaluating  $h_\mu h_\nu$  at  $\rho=c_1$ , it follows that

$$\frac{dS(\mathbf{r}')}{\sqrt{c_1^2 - \mu^2} \sqrt{c_1^2 - \nu^2}} = dS_e, \quad (\text{A } 7)$$

where  $dS_e$  is defined in equation (5.8).

By making use of equation (5.9), it follows that the curly bracket of the first term of the right-hand side of equation (2.17) equals

$$\sum_{n=1}^{\infty} \sum_{m=1}^{2n+1} \frac{1}{\gamma_n^m} C_n^m \mathbb{E}_n^m(\tau) E_n^m(M) E_n^m(N),$$

$$C_n^m = \sigma_c C_{cn}^{cm} - \sigma_f C_{fn}^{cm}. \quad (\text{A } 8)$$

Furthermore, for  $\mathbf{r}' \in \partial\Omega_c$ ,

$$\sum_{j=1}^3 \frac{x'_j \hat{\mathbf{x}}_j}{c_j} \times \nabla_{\mathbf{r}'} \frac{1}{|\mathbf{r} - \mathbf{r}'|}$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{2n+1} \frac{4\pi}{(2n+1)\gamma_n^m} \mathbb{F}_n^m(\mathbf{r}) \left( \frac{x'_1}{c_1} \hat{\mathbf{x}}_1 + \frac{x'_2}{c_2} \hat{\mathbf{x}}_2 + \frac{x'_3}{c_3} \hat{\mathbf{x}}_3 \right) \times \nabla \mathbb{E}_n^m(\mathbf{r}'). \quad (\text{A } 9)$$

The right-hand side of equation (A 9) can be further simplified by using the following remarkable formula derived in Dassios & Fokas (preprint c):

$$\left( \frac{x'_1}{c_1} \hat{\mathbf{x}}_1 + \frac{x'_2}{c_2} \hat{\mathbf{x}}_2 + \frac{x'_3}{c_3} \hat{\mathbf{x}}_3 \right) \times \nabla \mathbb{E}_n^m(\mathbf{r}')$$

$$= \sum_{l=1, l \neq m}^{2n+1} (c_{1n}^{ml} \hat{\mathbf{x}}_1 + c_{2n}^{ml} \hat{\mathbf{x}}_2 + c_{3n}^{ml} \hat{\mathbf{x}}_3) E_n^l(M) E_n^l(N),$$

$$\mathbf{r}' \in \partial\Omega_c. \quad (\text{A } 10)$$

Substituting this formula in the right-hand side of equation (A 9) and employing the orthogonality of the ellipsoidal harmonics, it follows that the first integral on the right-hand side of equation (2.17) equals

$$c_1 c_2 c_3 4\pi \sum_{n=1}^{\infty} \sum_{m=1}^{2n+1} \frac{\mathbb{F}_n^m(\mathbf{r})}{(2n+1)\gamma_n^m} C_n^m$$

$$\times \sum_{l=1, l \neq m}^{2n+1} (c_{1n}^{ml} \hat{\mathbf{x}}_1 + c_{2n}^{ml} \hat{\mathbf{x}}_2 + c_{3n}^{ml} \hat{\mathbf{x}}_3) E_n^l(\tau).$$

The evaluation of the other three integrals is similar. Hence, equation (5.10) follows, where the Cartesian components of the constant vector  $\mathbf{H}_n^{ml}$  are given by

$$H_{jn}^{ml} = c_1 c_2 c_3 [\sigma_c C_{cn}^{cm} - \sigma_f C_{fn}^{cm}] c_{jn}^{ml}$$

$$+ f_1 f_2 f_3 [\sigma_f C_{fm}^{fn} - \sigma_b C_{bm}^{fn}] f_{jn}^{ml}$$

$$+ b_1 b_2 b_3 [\sigma_b C_{bn}^{bm} - \sigma_s C_{sn}^{bm}] b_{jn}^{ml}$$

$$+ s_1 s_2 s_3 \sigma_s C_{sm}^{sn} s_{jn}^{ml}, \quad j = 1, 2, 3, \quad (\text{A } 11)$$

where  $C_{sm}^{ln}$  are defined in equations (A 1)–(A 5),  $c_{jn}^{ml}$ ,  $j=1, 2, 3$  are defined in equation (A 10) and  $(f_{jn}^{ml}, b_{jn}^{ml}, s_{jn}^{ml})$  are defined by equations similar to equation (A 10) with  $\mathbf{r}'$  on  $\partial\Omega_f$ ,  $\partial\Omega_b$ ,  $\partial\Omega_s$ , instead of  $\mathbf{r}'$  on  $\partial\Omega_c$ .

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